## **ALGEBRAIC INTEGER RELATIONS**

#### Dr. Matthew P. Skerritt



IPB University Mathematics Webinar Series 19th March 2021

## COMPUTATION OF INTEGER AND ALGEBRAIC INTEGER RELATIONS

Dr. Matthew P. Skerritt



IPB University Mathematics Webinar Series 19th March 2021

1. Introduction to Integer Relations

Introduction to Integer Relations
Computation of Classical Integer Relations

Introduction to Integer Relations
Computation of Classical Integer Relations
A Brief Introduction to Algebraic Number Theory

1. Introduction to Integer Relations

- 2. Computation of Classical Integer Relations
- 3. A Brief Introduction to Algebraic Number Theory
- 4. Algebraic Integer Relations

- 1. Introduction to Integer Relations
- 2. Computation of Classical Integer Relations
- 3. A Brief Introduction to Algebraic Number Theory
- 4. Algebraic Integer Relations
- 5. Computing Quadratic Integer Relations

1. Introduction to Integer Relations

#### Definition

Let  $x_1, \ldots, x_n \in \mathbb{R}$ . Any *n* integers,  $a_1, \ldots, a_n \in \mathbb{Z}$ are an *integer relation* of the original numbers  $x_i$  if

$$a_1x_1 + \dots + a_nx_n = 0$$

For convenience, we will usually think of  $x \in \mathbb{R}^n$  and  $a \in \mathbb{Z}^n$  and say that a is an integer realtion of x.



Aside: picture proof that  $\arctan 1 + \arctan 2 + \arctan 3 = \pi$ and so (4, -3, 4) really is an integer relation for  $(\arctan 2, \pi, \arctan 3)$ .

## HISTORY

Euclid's extended algorithm.

- a m + b n = gcd(a, b) relates a, b, and gcd(a, b) by integers.
- Each iteration produces incrementally better approximation.
- Also works for real numbers (see Book X, Prop 3).

## HISTORY

General case (for more than 2 numbers).

- Sought after by: Euler, Jacobi, Pincaré, Minkowski, Perron, Brun, Mahler, and others.
- None proved to work for n > 3
- None adequately captured the increasing approximation property.

## HISTORY

Generalisations satisfying both properties.

- First one published by Ferguson & Forcade in 1979 with extensive, unpublished notes, by Bergman.
- Several intermediate results (Ferguson & Forcade 1982, Ferguson 1986, Ferguson 1987).
- PSOS algorithm (Ferguson 1988)
- PSLQ algorithm (Ferguson & Bailey 1992)

Can see if a number  $\alpha$  satisfies an integer polynomial by searching for integer relations of  $(\alpha^0, \alpha^1, \dots, \alpha^n)$ 

Bailey & Ferguson used such a technique to show  $\gamma$ ,  $\log \gamma$ ,  $\log \pi$ ,  $\zeta(3)$  and some others do not satisfy a low-degree polynomial.

Borwein Bailey Plouffe formula

$$\pi = \sum_{k=0}^{\infty} \left( \frac{1}{16^k} \left( \frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right) \right)$$

found by "inspired guessing and extensive searching using the PSLQ integer relation algorithm"

Bailey & Plouffe use PSLQ to search for ("recognise") symbolic representations of floating point numbers.

E.g., they were able to recognise that

$$1.6083594219855456... \approx \int_0^\infty t^{7/4} e^{-t} dt = \frac{21\pi\sqrt{2}}{16\Gamma(1/4)}$$

(note that the RHS can be reduced to  $\frac{21}{16} \Gamma(3/4)$ )

Chamberland uses PSLQ to find relationships between functions.

There are many others. See Borwein & Lisoněk 2000 "Applications of Integer Relation Algorithms" (DOI: 10.1016/S0012-365X(99) 00256-3).

#### We can extend the idea to allow complex numbers.

#### Definition

Let  $x_1, \ldots, x_n \in \mathbb{R}$ . Integers,  $a_1, \ldots, a_n \in \mathbb{Z}$  are an *integer relation* of the original numbers  $x_i$  if

$$a_1x_1 + \dots + a_nx_n = 0$$

#### Definition

Let  $x_1, \ldots, x_n \in \mathbb{C}$ . Integers,  $a_1, \ldots, a_n \in \mathbb{Z}[i]$  are an *integer relation* of the original numbers  $x_i$  if

$$a_1x_1 + \dots + a_nx_n = 0$$

#### Note that $\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}$

This definition encapsulates the previous one.

- 1. 🔽 Introduction to Integer Relations
- 2. Computation of Classical Integer Relations
- 3. A Brief Introduction to Algebraic Number Theory
- 4. Algebraic Integer Relations
- 5. Computing Quadratic Integer Relations

#### 2. Computation of Classical Integer Relations

When talking about integer relations in general, we use the following notation so as to encapsulate all cases.

Notation

We will use  $\mathbb{F}$  to denote the field of numbers which we are relating and we will use  $\mathcal{O}$  to denote the ring of integers from which the relation elements are from. When talking about integer relations in general, we use the following notation so as to encapsulate all cases.

Notation

We will use  $\mathbb{F}$  to denote the field of numbers which we are relating and we will use  $\mathcal{O}$  to denote the ring of integers from which the relation elements are from.

The classical cases, then, are when  $\mathcal{O} = \mathbb{Z}$  and  $\mathbb{F} = \mathbb{R}$ , and when  $\mathcal{O} = \mathbb{Z}[i]$  and  $\mathbb{F} = \mathbb{C}$ .

#### Definition

Let  $B = \{b_1, \dots, b_n\} \subset \mathbb{F}^n$  be a linearly independent set of vectors. The set

$$L = \{\lambda_1 b_1, \ldots, \lambda_n b_n : \lambda_1 \in \mathcal{O}\}$$

is the lattice spanned by B. We call B the basis of the lattice L

We will sometimes refer to an  $\mathcal{O}$ -lattice if we need to specify the integer ring explicitly.

# We can represent a lattice L as a matrix by using the basis vectors as row vectors.

#### Example

The lattice spanned by the vectors (1, 2, 0) and (3, 0, 4), and (0, 5, 6) is given by the matrix

$$\begin{bmatrix} 1 & 2 & 0 \\ 3 & 0 & 4 \\ 0 & 5 & 6 \end{bmatrix}$$

# We can represent a lattice L as a matrix by using the basis vectors as row vectors.

#### Example

The lattice spanned by the vectors (1, 2, 0) and (3, 0, 4), and (0, 5, 6) is given by the matrix

$$\begin{bmatrix} 1 & 2 & 0 \\ 3 & 0 & 4 \\ 0 & 5 & 6 \end{bmatrix}$$

Note that the literature often uses column vectors instead of row vectors.

An information theoretic argument shows that in order to numerically compute an integer relation on *n* numbers, where the relation consists of integers with no more than *d* digits, then at least *nd* decimal digits of precision need to be used.

In the case of complex numbers and Gaussian integers, the number of digits is the number of digits in the real part, or the imaginary part, whichever is larger.

The LLL algorithm was introduced by Lenstra, Lenstra Jr, and Lovasz in 1982.

It is not an algorithm for computing integer relations directly. Instead it finds short lattice vectors

It can, however, be used to compute integer relations

Given a basis B for a lattice L, the LLL algorithm computes a basis  $B' = (b'_1, \ldots, b'_n)$  such that

$$|\mu_{i,j}| \le \frac{1}{2} \text{ for } 1 \le j < i \le n$$
$$||b_i^* + \mu_{i,j}b_{i-1}^*||^2 \ge \frac{3}{4}||b_{i-1}^*||^2 \text{ for } 1 < i \le n$$

where 
$$b_i^* = b'_i - \sum_{j=1}^{i-1} \mu_{i,j} b_j^*$$
 and  $\mu_{i,j} = \frac{b'_i \cdot b_j^*}{||b_j^*||^2}$ 

To find an integer relation on  $x = (x_1, ..., x_n) \in \mathbb{R}^n$ we can use LLL on the lattice



for some  $N \in \mathbb{R}$ .

Elements of the lattice look like

$$(\lambda_1,\ldots,\lambda_n,N(\lambda_1x_1+\cdots+\lambda_nx_n))$$

So if the  $\lambda_i$  are an integer relation for x then the last term will be 0.

We can prove that if N is large enough, then the first element of the reduced lattice must contain the integer relation.
# LLL

For the complex case where n  $z = (z_1, ..., z_n) \in \mathbb{C}^n$ we still use the augmented lattice  $L := [I_n Nz]$  like we did for the real case.

However we then construct a 2nd augmented lattice

$$egin{bmatrix} \Re L & \Im L \ -\Im L & \Re L \end{bmatrix}$$

and use LLL on that lattice.

# LLL

In all cases, we don't know how large to make N before we start. So we must use trial and error.

We have found, experimentally, that with numeric (floating point) calculations it is possible for N to be too large, even though the theory requires N to be "sufficiently large".

The PSLQ algorithm was introduced by Ferguson & Bailey in 1988. It was later analysed by Ferguson, Bailey, and Arno in 1999

It can handle both the real and complex cases directly.

We will not discuss the details of the algorithm itself. However we will establish some facts.

Observe that in both the real and complex cases  $\mathcal{O}$  forms a lattice, and that  $\mathcal{O} \subset \mathbb{F}$ .

Definition

Let  $x \in \mathbb{F}$ . An integer  $a \in \mathcal{O}$  is a *nearest integer* if |x - a| is maximal. A function  $[\cdot] : \mathbb{F} \to \mathcal{O}$  is a nearest integer functino if it maps each  $x \in \mathbb{F}$  to one of its nearest integers.

We denote by  $\lceil \cdot \rfloor_{\mathcal{O}}$  if we need to be specific.

There are parameters:  $\gamma$ ,  $\rho$ , and  $\tau$  that must satisfy

$$\frac{1}{\rho} \ge |x - \lceil x \rfloor| \text{ for all } x \in \mathbb{F}$$

$$1 \le \tau \le \rho$$

$$\frac{1}{\tau^2} = \frac{1}{\gamma^2} + \frac{1}{\rho^2}$$

 $\rho$  is bounded by the integer lattice; choose to maximise  $1/\rho.$ 

- $\gamma$  is minimised when  $\tau = 1$ ; choose any larger value.
- au is fixed by the choice of  $\gamma$  and the value of  $\rho$ .

Each iteration of the PSLQ algorithm finds an increased lower bound on the norm of *any* integer relation on the input.

The algorithm can terminate when that bound exceeds some threshold.

If the algorithm finds a relation, a then  $||a|| \leq \gamma^{n-2} M$  where M is the norm of the smallest possible relation.

#### LLL requires more precision than PSLQ



**Real Integer Relation Problems** 

#### LLL requires more precision than PSLQ



Complex Integer Relation Problems (real constants)

#### LLL requires more precision than PSLQ



Complex Integer Relation Problems (complex constants)

#### PSLQ Required Precision (as % of theoretical min).



**Real Integer Relation Problems** 

#### PSLQ Required Precision (as % of theoretical min).



Complex Integer Relation Problems (real constants)

#### PSLQ Required Precision (as % of theoretical min).



Complex Integer Relation Problems (complex constants)

#### LLL takes much more time than PSLQ



**Real Integer Relation Problems** 

#### LLL takes much more time than PSLQ



Complex Integer Relation Problems (real constants)

#### LLL takes much more time than PSLQ



Complex Integer Relation Problems (complex constants)

# TALK SUMMARY

- 1. VIII Introduction to Integer Relations
- 2. 🗹 Computation of Classical Integer Relations
- 3. A Brief Introduction to Algebraic Number Theory
- 4. Algebraic Integer Relations
- 5. Computing Quadratic Integer Relations

3. A Brief Introduction to Algebraic Number Theory

Let  $\mathbb{F}$  be a field. If  $\mathbb{F}$  is a subfield of some other field  $\mathbb{K}$  (i.e.,  $\mathbb{F} \subseteq \mathbb{K}$  then we call  $\mathbb{K}$  a *field extension* (or, equivalently an *extension field*) of  $\mathbb{F}$ .

We denote this by  $\mathbb{K}$  :  $\mathbb{F}$  when we need to be explicit about the base field and the extension.

Note that the literature often uses the notation  $\mathbb{K}/\mathbb{F}$  instead.

Let  $\mathbb{F}$  be a field, and consider an extension field  $\mathbb{K}$  :  $\mathbb{F}$ . An element  $k \in \mathbb{K}$  is said to be *algebraic over*  $\mathbb{F}$  if it satisfies

$$f_n k^n + \dots + f_1 k + f_0 = 0$$

for some  $n \in \mathbb{N}$  and  $f_i \in \mathbb{F}$ . Note that  $f_n \neq 0$ .

For example  $\sqrt{2}$  satisfies  $k^2 - 2 = 0$  and so is algebraic over the rationals.

Let D be an integral domain and consider an extension field  $\mathbb{K}$ :  $\mathbb{F}$  such that  $D \subseteq \mathbb{K}$ . An element  $k \in \mathbb{K}$  is said to be *integeral over*  $\mathbb{F}$  if it satisfies a monic polynomial

$$k^{n} + d_{n-1}k^{n-1} + \dots + d_{1}k + d_{0} = 0$$

for some  $n \in \mathbb{N}$  and  $d_i \in D$ 

An integral domain is a commutative ring with multiplicative identity and no non-zero elements  $d_1$ ,  $d_2$  with the property that  $d_1d_2 = 0$ .

#### Let $z \in \mathbb{C}$ .

We denote by  $\mathbb{Q}(z)$  the smallest subfield of  $\mathbb{C}$  that contains both  $\mathbb{Q}$  and z.

We denote by  $\mathbb{Z}[z]$  the smallest subring of  $\mathbb{C}$  that contains both  $\mathbb{Z}$  and z.

Note that the ring  $\mathbb{Z}$  is often called the "rational integers" to avoid ambiguity.

We are primarily interested in extension fields of  $\mathbb{Q}$ .

#### Definition

A number  $z \in \mathbb{C}$  is an *algebraic number* (or simply *algebraic*) if it is algebraic over  $\mathbb{Q}$ .

A number  $z \in \mathbb{C}$  is an *algebraic integer* if it is integral over  $\mathbb{Z}$ .

The ring of all algebraic integers is denoted by  $\mathcal{A}$ .

An algebraic extension is *real* if  $\mathbb{K} \subseteq \mathbb{R}$  and *complex* otherwise.

We are primarily interested in extension fields of  $\mathbb{Q}$ .

#### Definition

An extension field  $\mathbb{K}$  :  $\mathbb{Q}$  is an *algebraic extension* field (or simply an *algebraic extension* if every  $k \in \mathbb{K}$  is an algebraic number.

Its ring of integers of  $\mathbb{K}$ , denoted  $\mathcal{O}_{\mathbb{K}}$ , is  $\mathbb{K} \cap \mathcal{A}$ . We call such rings algebraic integers

#### We are primarily interested in extension fields of $\mathbb{Q}$ .

#### Example

The set  $\mathbb{Q}(\sqrt{5})$  is an algebraic extension field. Its elements look like  $\{q_1 + q_2\sqrt{5} : q_i \in \mathbb{Q}\}$ , such as  $\frac{1}{2} + \frac{32}{101}\sqrt{5}$ 

The golden ratio  $\frac{1+\sqrt{5}}{2}$  is an algebraic integer of  $\mathbb{Q}(\sqrt{5})$  because it satisfies the polynomial  $x^2 - x - 1 = 0$ . Similarly for  $\frac{1-\sqrt{5}}{2}$ .

We will restrict our attention to quadratic extension fields. These are fields of the form:

$$\mathbb{Q}(\sqrt{D}) = \left\{ q_1 + q_2 \sqrt{D} : q_i \in \mathbb{Q} \right\}$$

wher  $D \in \mathbb{Z}$  is square free.

Elements of these fields satisfy *quadratic* (or lower degree) polynomials with rational coefficients.

The algebraic integers of a quadratic field (which we call *quadratic integers*) are of the form

$$\mathcal{O}_{\mathbb{Q}(\sqrt{D})} = \mathbb{Z}[\omega] = \{m_1 + m_2\omega : m_i \in \mathbb{Z}\}$$

where

$$\omega = \begin{cases} \sqrt{D} & \text{if } D \equiv 2,3 \pmod{4} \\ \frac{1+\sqrt{D}}{2} & \text{if } D \equiv 1 \pmod{4} \end{cases}$$



 $D \equiv 2, 3 \pmod{4}$ 



 $D \equiv 1 \pmod{4}$ 

Finally we mention a special class of quadratic fields.

Real norm Euclidean fields

 $\begin{array}{l} \mathbb{Q}(\sqrt{2}), \mathbb{Q}(\sqrt{3}), \mathbb{Q}(\sqrt{5}), \mathbb{Q}(\sqrt{6}), \mathbb{Q}(\sqrt{7}), \mathbb{Q}(\sqrt{11}), \\ \mathbb{Q}(\sqrt{13}), \mathbb{Q}(\sqrt{17}), \mathbb{Q}(\sqrt{19}), \mathbb{Q}(\sqrt{21}), \mathbb{Q}(\sqrt{29}), \\ \mathbb{Q}(\sqrt{33}), \mathbb{Q}(\sqrt{37}), \mathbb{Q}(\sqrt{41}), \mathbb{Q}(\sqrt{55}), \mathbb{Q}(\sqrt{73}) \end{array}$ 

Complex norm Euclidean fields  $\mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-2}), \mathbb{Q}(\sqrt{-3}), \mathbb{Q}(\sqrt{-7}), \mathbb{Q}(\sqrt{-11})$ 

# TALK SUMMARY

- 1. VIII Introduction to Integer Relations
- 2. 🗹 Computation of Classical Integer Relations
- 3. 🗹 A Brief Introduction to Algebraic Number Theory
- 4. Algebraic Integer Relations
- 5. Computing Quadratic Integer Relations

4. Algebraic Integer Relations
Recall the definiton of an integer relation:

#### Definition

Let  $x_1, \ldots, x_n \in \mathbb{C}$ . Integers,  $a_1, \ldots, a_n \in \mathbb{Z}[i]$ are an *integer relation* of the original numbers  $x_i$  if

$$a_1x_1 + \dots + a_nx_n = 0$$

#### Definition

Let  $x_1, \ldots, x_n \in \mathbb{C}$ . Integers,  $a_1, \ldots, a_n \in \mathbb{Z}[i]$ are an *integer relation* of the original numbers  $x_i$  if

$$a_1x_1 + \dots + a_nx_n = 0$$

We could try and replace  $\mathbb{C}$  with an algebraic extension field, and replace  $\mathbb{Z}[i]$  with the ring of integers of that field.

We could try and replace  $\mathbb{C}$  with an algebraic extension field, and replace  $\mathbb{Z}[i]$  with the ring of integers of that field.

However, this won't capture the existing cases.

The fields  $\mathbb{R}$  and  $\mathbb{C}$  are not algebraic fields (although they are extension fields of  $\mathbb{Q}$ ). Thus they do not have algebraic integers.

If we try to take the intersections anyway ( $\mathbb{R} \cap \mathcal{A}$  or  $\mathbb{C} \cap \mathcal{A}$ ) we do not get correct integers.

Instead, specify an intermediate extension field.

#### Definition

Let  $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$  and let  $\mathbb{K}$  be an algebraic extension field. Let  $\mathcal{O} = \mathcal{O}_{\mathbb{K}}$ .

Let  $x_1, \ldots, x_n \in \mathbb{F}$ . Integers,  $a_1, \ldots, a_n \in \mathcal{O}$  are an *algebraic integer relation* (or a K-integer relation) of the original numbers  $x_i$  if

$$a_1x_1 + \dots + a_nx_n = 0$$

# TALK SUMMARY

- 1. VI Introduction to Integer Relations
- 2. 🗹 Computation of Classical Integer Relations
- 3. 🗹 A Brief Introduction to Algebraic Number Theory
- 4. 🗹 Algebraic Integer Relations
- 5. Computing Quadratic Integer Relations

5. Computing Quadratic Integer Relations

Observe that for an quadratic integer relation:

$$\begin{pmatrix} m_{1,1} + m_{1,2}\omega \end{pmatrix} x_1 + \dots + \begin{pmatrix} m_{n,1} + m_{n,2}\omega \end{pmatrix} x_n = \begin{pmatrix} m_{1,1}x_1 + \dots + m_{n,1}x_n \end{pmatrix} + \begin{pmatrix} m_{1,2}\omega x_1 + \dots + m_{n,2}\omega x_n \end{pmatrix}$$

This suggests a way to reduce a quadratic integer relation to a classical one.

Given  $(x_1, \ldots, x_n) \in \mathbb{F}^n$  we compute  $(x_1, \ldots, x_n, x_1\omega, \ldots, x_n\omega) \in \mathbb{F}^{2n}$  and search for a classical integer relation on that.

We recover the integer relation by taking  $a_k + a_{n+k}\omega$ for  $1 \le k \le n$ .

This works well for real quadratic fields (i.e., when D > 1). It works less well for complex quadratic fields.

In the complex case, the integers are of the form  $a_k = m_{k,1} + m_{k,2}i$ . So when we attempt to recover the relation we get

$$(m_{k,1}+m_{k,2}i)+(m_{k+n,1}+m_{k+n,2}i)\omega$$

after expansion and some simplification we get

$$\alpha_1 + \beta_2 i \sqrt{|D|} + \alpha_2 i - \beta_2 \sqrt{|D|}$$
 for  $\alpha_1, \beta_1 \in \mathbb{Z}$ 

Much of the time the reduction technique works fine, and gives good answers even in the complex cases.

When it does not work, we try two techniques to nonetheless extract a relation.

Decomposition method

We observe that

$$\alpha_1 + \beta_2 i \sqrt{|D|} + \alpha_2 i - \beta_2 \sqrt{|D|}$$

can be written as

$$(\alpha_1 + \beta_2 i \sqrt{|D|}) + i (\alpha_2 + \beta_2 \sqrt{|D|})$$

**Conjugate Method** 

Given

$$\alpha_1 + \beta_2 i \sqrt{|D|} + \alpha_2 i - \beta_2 \sqrt{|D|}$$

a conjugate is

$$\alpha_1 - \beta_2 i \sqrt{|D|} - \alpha_2 i - \beta_2 \sqrt{|D|}$$

Multiplying the number by its conjugate gives an element of  $\mathcal{O}_{\mathbb{Q}(\sqrt{D})}$ . We multiply all elements of the relation by the conjugate of only one of the elements.

Both the methods for correcting the complex case work well in practice.

We do not have proofs that they will always be correct. It is nonetheless easy to just try them, and check the extracted relation.

#### **USING LLL**

To compute an algebraic integer n  $z = (z_1, ..., z_n) \in \mathbb{F}^n$  we still use the augmented lattice  $L := [I_n Nz]$  like we did for the real case.

However we then construct a 2nd augmented lattice.

 $\begin{bmatrix} \mathfrak{R}L & \mathfrak{I}L \\ \mathfrak{W}_1L & \mathfrak{W}_2L \end{bmatrix}$ 

and use LLL on that lattice.

# **USING LLL**

However we then construct a 2nd augmented lattice.

 $egin{bmatrix} {\mathfrak{R}}L & {\mathfrak{T}}L \ {\mathfrak{W}}_1L & {\mathfrak{W}}_2L \end{bmatrix}$ 

and use LLL on that lattice.

 $\mathfrak{W}_{1}x_{k} = \begin{cases} -\sqrt{|D|}\mathfrak{T}x_{k} & \text{if if } D \equiv 2,3 \pmod{4} \\ (\mathfrak{R}x_{k} - \sqrt{|D|}\mathfrak{T}x_{k})/2 & \text{if } D \equiv 1 \pmod{4} \end{cases}$ 

 $\mathfrak{W}_{1}x_{k} = \begin{cases} -\sqrt{|D|}\mathfrak{T}x_{k} & \text{if if } D \equiv 2,3 \pmod{4} \\ (\mathfrak{R}x_{k} - \sqrt{|D|}\mathfrak{T}x_{k})/2 & \text{if } D \equiv 1 \pmod{4} \end{cases}$ 

# USING LLL

As with the other cases, so long as N is large enough then the procedure must find an algebraic integer relation.

We attempt to modify PSLQ while keeping the existing theory.

We simply use a quadratic integer nearest integer function, and modify the algorithm to allow us to specify the quadratic extension field we wish to use.

We simply use a quadratic integer nearest integer function, and modify the algorithm to allow us to specify the quadratic extension field we wish to use.

This introduces a problem: for real quadratic extension fields, the integers do not form a lattice (there is no unique nearest integer to a field element), so we cannot use the existing theory.

We restrict our attention to the complex cases, then.

We restrict our attention to the complex cases, then.

In these cases we have a more subtle problem.

As D increases, the parameter  $\rho$  decreases, until eventually  $\rho < 1$  making it impossible to satisfy the  $1 < \tau \leq \rho$  condition.

In these cases we have a more subtle problem.

As D increases, the parameter  $\rho$  decreases, until eventually  $\rho < 1$  making it impossible to satisfy the  $1 < \tau \leq \rho$  condition.

The complex quadratic fields that do not exhibit these problems are exactly:

$$\mathbb{Q}(\sqrt{-2}), \mathbb{Q}(\sqrt{-3}), \mathbb{Q}(\sqrt{-7}), \mathbb{Q}(\sqrt{-11})$$

As D increases, the parameter  $\rho$  decreases, until eventually  $\rho < 1$  making it impossible to satisfy the  $1 < \tau \leq \rho$  condition.

The complex quadratic fields that do not exhibit these problems are exactly:

$$\mathbb{Q}(\sqrt{-2}), \mathbb{Q}(\sqrt{-3}), \mathbb{Q}(\sqrt{-7}), \mathbb{Q}(\sqrt{-11})$$

These are precisely the complex quadratic norm Euclidean fields.

The complex quadratic fields that do not exhibit these problems are exactly:

$$\mathbb{Q}(\sqrt{-2}), \mathbb{Q}(\sqrt{-3}), \mathbb{Q}(\sqrt{-7}), \mathbb{Q}(\sqrt{-11})$$

We note that we've had surprising success with the modified PSLQ on

$$\mathbb{Q}(\sqrt{-5}), \mathbb{Q}(\sqrt{-6}), \mathbb{Q}(\sqrt{-10})$$

## TALK SUMMARY

✓ Introduction to Integer Relations
✓ Computation of Classical Integer Relations
✓ A Brief Introduction to Algebraic Number Theory
✓ Algebraic Integer Relations
✓ Computing Quadratic Integer Relations

# THANKYOU!